Chapter 15

Rivers and Streams

SUMMARY: Rivers are, in first approximation, nearly one-dimensional flows driven by gravity down a slope and resisted by friction. While this may seem simple from a physical perspective, nonlinearities in the dynamics engender complex behavior. After a description of the basic hydraulic regimes, the chapter addresses water-quality issues.

15.1 Open Channel Flow

Introduction

Streams and rivers form an essential link in the hydrological cycle and, in that capacity, provide freshwater for consumption and irrigation. Through their watershed, they also gather, convey and disperse almost any substance that enters water on land. Streams and rivers are thus central actors of environmental transport and fate.

Rivers and streams are types of open channels, i.e., conduits of water with a free surface. In contrast to canals, ditches, aquaducts and other structures designed and built by humans, rivers and streams are the products of natural geological processes and, as a consequence, are quite irregular. They have the ability to scour their beds, carry sediments and deposit these sediments, forever altering their own channels. Although there is no precise distinction made between rivers and streams, streams (Figure 15.1) are smaller and more rugged, their depth is shallow, and their waters generally flow faster, whereas rivers (Figure 15.2) are deeper, wider and more tranquil. As we shall see, open-channel dynamics, called *hydraulics*, allow for two rather opposite types of motion, one shallow and fast (supercritical) and the other deep and slow (subcritical). However, it would be unwise to charaterize streams by one type and rivers by the other, because the same channel may exhibit varying properties

Figure 15.1: A small stream in Vermont, USA. [Photo by the author]

Figure 15.2: The Connecticut River at the level of Hanover, New Hampshire, USA. [Photo by the author]

along its downstream path that make the water alternatively pass from one regime to the other. This is often the case when the bottom slope is irregular.

In a first step, we establish the equations governing the water velocity and water depth as functions of the downstream distance and time, with particular attention paid to the case of a rectangular channel bed. Then, we consider a series of particular cases of interest: steady and unsteady flow, gradually and rapidly varying in the downstream direction.

Equations of motion

River flow is actually three-dimensional because the velocity depends not only on downstream distance but also on depth and transverse position. This is so because friction against the bottom and banks causes the velocity to decrease from a maximum at the surface near the middle of the stream to zero along the bottom and sides. In addition, centrifugal effects in river bends generate secondary circulations that render the velocity a full three-dimensional vector.

Because we wish to emphasize here the manner by which the flow varies in the downstream direction, we will neglect cross-stream velocity components as well as cross-stream variations of the downstream component, by considering the speed u as the water velocity averaged across the stream and a function of only the downstream distance x and time t. Because the flow in a river almost never reverses, the fact that we take x directed downstream implies that u is a positive quantity.

With a free surface exposed to the atmosphere, the water depth in a river can, too, vary in space and time. This implicates a second flow variable, namely the water depth, which we denote h and take as function of x and t, like the velocity. And like u, h must be positive everywhere. The existence of two dependent variables, $u(x,t)$ and $h(x,t)$, calls for two governing equations. Naturally, these are statements of mass conservation and momentum budget.

To establish the pair of governing equations, consider a slice of river as depicted in Figure 15.3. Geometric quantities are: A the cross-sectional area occupied by the water, P the *wetted perimeter* (shortest underwater distance from bank to opposite bank following the curved bottom), $S = \sin\theta$ the bottom slope, and h the water depth at the deepest point. The cross-sectional area A and wetted perimeter P are each a function of the water depth h, because as h rises A and P increase in a way that depends on the shape of the channel bed. For example, a channel bed with rectangular cross-section of width W (Figure 15.4) yields $A = Wh$ and $P = W + 2h$.

Mass conservation

Conservation of mass is relatively straightforward. We simply need to state that the accumulation over time of mass $\rho A dx$ inside the slice of length dx is caused by a possible difference between the amount of mass $\rho A u$ that enters per time at position x and the amount that leaves per time at position $x + dx$. For a short time interval

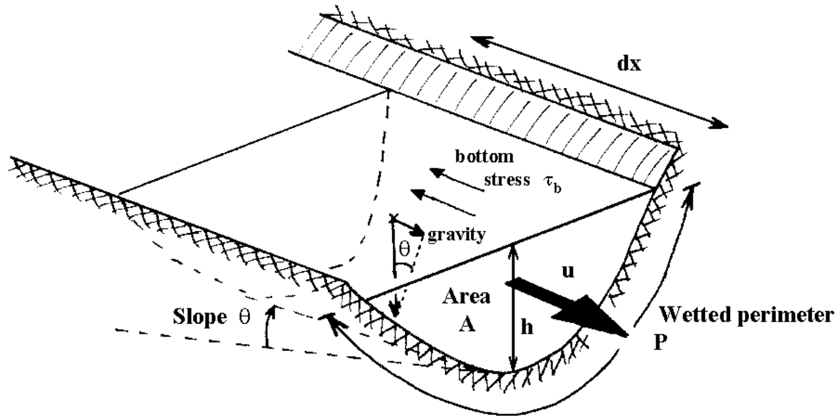

Figure 15.3: A slice of length dx along a river for the formulation of mass conservation and momentum budget. The notation is: velocity averaged across the stream u, water depth h, cross-sectional area of the stream A, wetted perimeter P, and bottom slope $S = \sin\theta$.

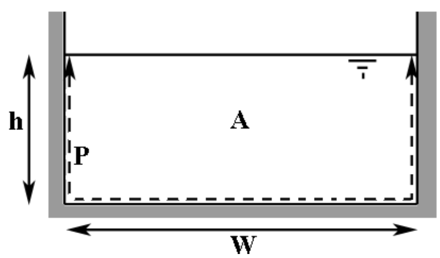

Figure 15.4: A channel bed with rectangular cross-section. In this simplest of cases, $A = Wh$ and $P = h + W + h = W + 2h$.

dt, this mass budget[1] is:

$$\begin{aligned} \rho A dx|_{\text{at } t+dt} &= \rho A dx|_{\text{at } t} \\ &+ \rho A u|_{\text{at } x} - \rho A u|_{\text{at } x+dx} \end{aligned}$$

which, in the limit of dt and dx going to zero, becomes:

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A u) = 0,$$

or, because water is incompressible (ρ constant):

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(A u) = 0. \tag{15.1}$$

[1]This equation is attributed to Leonardo da Vinci (1452–1519), although he did not write it in terms of derivatives.

Since the manner in which the cross-sectional area A increases with the water depth h is known from the shape of the channel bed, the preceding equation actually governs the temporal evolution of the water depth h. It requires the knowledge of the velocity u, for which a second equation is necessary. This will be fulfilled once we have established the momentum budget.

In the meantime, it is instructive to write the mass-conservation equation in the case of a rectangular cross-section of constant width. With $A = Wh$, Equation (15.1) reduces to:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0. \tag{15.2}$$

Momentum budget

We write that the time rate of change of momentum inside our slice of river is the momentum flux entering upstream, minus the momentum flux exiting downstream, plus the sum of accelerating forces (acting in the direction of the flow), and minus the sum of the decelerating forces (acting against the flow). Symbolically:

$$\begin{aligned} \frac{d}{dt}\,[\text{Momentum inside the slice}] \;=\;& \text{Momentum flux entering at } x \\ -\;& \text{Momentum flux exiting at } x + dx \\ +\;& \text{Pressure force in the rear} \\ -\;& \text{Pressure force ahead} \\ +\;& \text{Downslope gravitational force} \\ -\;& \text{Frictional force along the bottom.} \end{aligned}$$

The momentum is the mass times the velocity, that is $(\rho dV)u = \rho A u dx$, whereas the momentum flux is the mass flux times the velocity, that is $(\rho A u)u = \rho A u^2$. The pressure force F_p at each end of the slice is obtained from the integration of the depth-dependent pressure over the cross-section:

$$\text{Pressure force} = F_p = \iint p\,dA = \int_0^h p(z) w(z) dz,$$

in which $p(z)$ and $w(z)$ are, respectively, the pressure and channel width at level z, with z varying from zero at the bottom-most point to h at the surface. Under the assumption of a hydrostatic balance, the pressure increases linearly with depth according to

$$p(z) = \rho g (h - z), \tag{15.3}$$

discounting the atmospheric pressure which acts all around and has no net effect on the flow. The pressure force is thus equal to:

$$F_p = \int_0^h \rho g (h - z) w(z) dz, \tag{15.4}$$

and is a function of how filled the channel is. In other words, it is a function of depth h. Taking the h derivative (which will be needed later), we have:

$$\begin{aligned}\frac{dF_h}{dh} &= [\rho g(z-h)w(z)]_{z=h} \;+\; \int_0^h \rho g w(z) dz \\ &= \rho g \int_0^h w(z) dz \;=\; \rho g A.\end{aligned} \tag{15.5}$$

The gravitational force is the weight of the water slice projected along the x–direction, which is mg times the sine of the slope angle θ:

$$\text{Gravitational force} \;=\; [(\rho dV)g]\sin\theta \;=\; \rho g A S dx. \tag{15.6}$$

Finally, the frictional force is the bottom stress τ_b multiplied by the wetted surface area:

$$\text{Frictional force} \;=\; \tau_b P dx. \tag{15.7}$$

River flows are typically in a state of turbulence and, within a certain level of approximation, the bottom stress is proportional to the square of the velocity. Invoking a drag coefficient C_D, we write:

$$\text{Bottom stress} \;=\; \tau_b \;=\; C_D \rho u^2, \tag{15.8}$$

which resembles a Reynolds stress ($\tau = -\rho\overline{u'w'}$, with the turbulent fluctuations u' and w' each proportional to the average velocity u). The frictional force exerted on the slice of water is then:

$$\text{Frictional force} \;=\; \tau_b P dx \;=\; C_D \rho P u^2 dx. \tag{15.9}$$

Values for the drag coefficient in rivers vary between 0.003 and 0.02, but there is no universal value for a given channel bed because C_D varies with the Reynolds number of the flow as well as with the shape and roughness of the channel bed. For the sake of mathematical simplicity, however, we do not enter into those details right away.

We now gather the pieces of the momentum budget:

$$\begin{aligned}\frac{[\rho A u dx|_{\text{at } t+dt} \;-\; \rho A u dx|_{\text{at } t}]}{dt} &= \rho A u^2|_{\text{at } x} \;-\; \rho A u^2|_{\text{at } x+dx} \\ &+ \; F_p|_{\text{at } x} \;-\; F_p|_{\text{at } x+dx} \\ &+ \; \rho g A S dx \\ &- \; C_D \rho P u^2 dx,\end{aligned}$$

or, in differential form,

$$\frac{\partial}{\partial t}(\rho A u) \;+\; \frac{\partial}{\partial x}(\rho A u^2) \;=\; -\;\frac{\partial F_p}{\partial x} \;+\; \rho g A S \;-\; C_D \rho P u^2.$$

Using the mass-conservation equation (15.1), we can reduce the left-hand side of this equation. Then, thanks to (15.5), the gradient of the pressure force becomes

$$\frac{\partial F_p}{\partial x} = \frac{dF_p}{dh}\frac{\partial h}{\partial x} = \rho g A \frac{\partial h}{\partial x} .$$

A division by ρA finally yields:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - g \frac{\partial h}{\partial x} + gS - C_D \frac{u^2}{R_h} . \tag{15.10}$$

In this equation the ratio of the cross-sectional area A over the wetted perimeter P, which has the dimension of a length, was defined as

$$R_h = \frac{A}{P} . \tag{15.11}$$

This is called the *hydraulic radius.* Because most rivers are much wider than they are deep, the wetted perimeter is generally not much more than the width ($P \simeq W$), and the hydraulic radius is approximately the average depth $\bar{h}$, which itself is not very different from the center depth if the channel has a broad flat bottom, as is often the case with natural streams:

$$R_h \simeq \frac{A}{W} = \bar{h} \simeq h. \tag{15.12}$$

The average depth $\bar{h}$ is exactly equal to the maximum depth h for a rectangular cross-section (Figure 15.4).

In Equation (15.10), the quantity R_h is a function of the water depth h. The momentum equation, therefore, establishes a new relation between the velocity u and depth h, which together with mass conservation (15.1) forms a closed set of two equations for two unknowns.

Because each equation contains a first-order derivative in time and also one in space, the system is of second order in both time and space. Two initial conditions and two boundary conditions are thus required to specify fully the problem. The initial conditions are naturally the spatial distribution of $h_o(x)$ and $u_o(x)$ at some original time, but it is far less clear what the boundary conditions ought to be and where they should be applied. As we shall see, imposing an upstream value of h and an upstream value of u does not necessarily work.

For a wide channel with broad flat bottom or with a rectangular cross-section, R_h may be replaced by h, and the momentum equation reduces to:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - g \frac{\partial h}{\partial x} + gS - C_D \frac{u^2}{h} . \tag{15.13}$$

The pair (15.1)–(15.10) are called the Saint-Venant[2] equations. Even in its simplified form, the set (15.2)–(15.13) for a wide rectangular channel is highly nonlinear. So, non-unique solutions and other surprises may occur.

[2]Adhémar de Saint-Venant (1797–1886), a French civil engineer who spent a significant part of his career working for the country's Bridges and Highways Department

15.2 Uniform Frictional Flow

Our first particular case is that of a steady and uniform flow down a constant slope. With the temporal and spatial derivatives set to zero, Equation (15.1) is trivially satisfied, while the momentum budget (15.10) reduces to:

$$C_D \frac{u^2}{R_h} = gS, \tag{15.14}$$

which simply states that the downslope force of gravity is resisted entirely by bottom friction. This is similar to a parachute in action, in which the downward force of gravity, which is constant, is balanced by the upward force of air drag, which is proportional to the square of the velocity. The equation can be readily solved for the velocity:

$$u = \sqrt{\frac{gR_hS}{C_D}} . \tag{15.15}$$

This is known as the Chézy[3] formula.

For a wide channel with broad flat bottom, the hydraulic radius R_h is nearly the water depth h, and (15.15) reduces to:

$$u = \sqrt{\frac{ghS}{C_D}} . \tag{15.16}$$

The formula (in either form) is physically correct but hides a complication inside the drag coeffcient C_D, which varies from river to river and with the water depth. Over the years, a number of improvements to the formula have been proposed to render the dependence on the water depth and bed roughness more explicit. We shall present only two here.

River flow falls in the category of shear turbulence. Thus, according to Section 8.2, an appropriate representation of the velocity profile over depth is the logarithmic profile:

$$u(z) = \frac{u_*}{\kappa} \ln \frac{z}{z_o} , \tag{15.17}$$

in which u_* is the friction velocity (related to the stress against the boundary), κ the von Kármán constant, and z_o the roughness height (a fraction of the mean height of the bottom asperities). As shown in Section 8.2, from this profile, one can obtain a relation between the bottom stress and the depth-averaged velocity u:

$$\tau_b = \frac{\rho\kappa^2 u^2}{[\ln(h/z_o) - 1]^2} . \tag{15.18}$$

from which follows an expression for the drag coefficient:

[3] in honor of Antoine Léonard de Chézy (1718–1798), a French engineer who designed canals for supplying water to the city of Paris

$$C_D = \frac{\kappa^2}{[\ln(h/z_o) - 1]^2} . \quad (15.19)$$

It is clear from this expression that the drag coefficient depends on both the roughness of the channel bed and the water depth. For $\kappa = 0.41$ and a height ratio h/z_o in the range 30–2000, the drag coefficient varies between 0.004 and 0.03. Substituting in the Chézy formula (15.16), we obtain:

$$u = \sqrt{\frac{ghS}{\kappa^2}} \left[\ln \frac{h}{z_o} - 1 \right] . \quad (15.20)$$

Having abundant data at his disposal and looking for a power law, Manning[4] determined that a 2/3 power was giving the best fit and proposed the following formula, in terms of the hydraulic radius:

$$u = \frac{1}{n} R_h^{2/3} S^{1/2} . \quad (15.21)$$

This is not too surprising since for realistic values of h/z_o (on the order of 1000), the best power-law fit to the function $\ln(h/z_o) - 1$ is $1.87(h/z_o)^{1/6}$, turning (15.20) into a 2/3 power of h.

In Equation(15.21), the coefficient n in the denominator is called the *Manning coefficient* and its value depends on the roughness of the channel bed (Table 15.1).

Regardless of what is done with the drag coeffcient, Equation (15.15) remains valid from basic physical principles. This gives the water velocity u in terms of the water depth h, and we may ask: How does a river select a specific value for u and a specific value for h water depth among the infinite possibilities offered by this functional relationship? The degree of freedom is set by the river's volumetric flow rate, called the *discharge* and noted Q. With

$$Q = Au, \quad (15.22)$$

Equations (15.15) and (15.22) form a two-by-two system of equations for h and u.

The solution in the particular case of a wide rectangular channel ($A = Wh$ and $R_h \simeq h$) is:

$$h = \left(\frac{C_D Q^2}{gSW^2} \right)^{\frac{1}{3}} \quad (15.23)$$

$$u = \left(\frac{gSQ}{C_D W} \right)^{\frac{1}{3}} . \quad (15.24)$$

[4] Robert Manning (1816–1897), Irish engineer and surveyor. History reveals that he was the first to propose a 2/3 power law.

Table 15.1: Values of the Manning coefficient for common channels

CHANNEL TYPE		n
Artificial channels	finished cement	0.012
	unfinished cement	0.014
	brick work	0.015
	rubble masonry	0.025
	smooth dirt	0.022
	gravel	0.025
	with weeds	0.030
	cobbles	0.035
Natural channels	mountain streams	0.045
	clean and straight	0.030
	clean and winding	0.040
	with weeds and stones	0.045
	most rivers	0.035
	with deep pools	0.040
	irregular sides	0.045
	dense side growth	0.080
Flood plains	farmland	0.035
	small brushes	0.125
	with trees	0.150

Because the water depth h varies like $Q^{2/3}$ whereas the velocity u varies as $Q^{1/3}$, we deduce that an increase in discharge generates a larger increase in depth than in velocity. So, when a flood condition arises, a river adapts by increasing its depth more than its velocity. The interesting result, however, is that the two quantities are intimately related to each other. It is presumed that this is the reason why Roman engineers of antiquity were successful at conveying clean water by aquaducts and removing waste water by sewers[5].

When the flow is not uniform but gradually varying, because the slope is not constant or there are other elements that activate the derivatives in (15.1) and (15.10), the value of h given by (15.23) is not necessarily the water depth realized by the stream but nonetheless serves as a useful reference against which the actual water depth may be compared. In this case, it is called the *normal depth* and is denoted by h_n:

$$h_n = \left(\frac{C_D Q^2}{g S W^2} \right)^{\frac{1}{3}} . \tag{15.25}$$

As we shall see later, the cases $h < h_n$ (flow is too thin and fast) and $h > h_n$ (the flow is too thick and slow) exhibit different dynamical properties.

[5]Indeed, Romans did not have a notion of time on the scale of the second and minute, only on the scale of hours and days by following the motion of the sun in the sky. As a consequence, they had no concept of a velocity and only had at their disposal the water depth, which they could measure with a stick. So, all their calculations were exclusively based on water depth, but the fact that h and u are tightly related to each other allowed them to obtain practical estimates for the design of their water lines.

15.3 The Froude Number

In what follows, a dimensionless ratio plays a crucial role. It is the so-called *Froude number*, defined from the average velocity u and the average depth $\bar{h}$ as:

$$Fr \;=\; \frac{u}{\sqrt{g\bar{h}}} \;. \tag{15.26}$$

Physically, it compares the actual water velocity to the speed of gravity waves on the surface (traveling at speed $\sqrt{gh}$ in a shallow fluid of depth h, according to Section 4.1.5).

Two cases arise. Either the water flows less fast than waves on its surface ($u < \sqrt{g\bar{h}} \rightarrow Fr < 1$) and the flow is said to be *subcritical*, or the water flows faster than the waves on its surface $u > \sqrt{g\bar{h}} \rightarrow Fr > 1$) and the flow is said to be *supercritical*. The flow is *critical* when its Froude number is unity.

The *critical depth* is the depth that a given discharge $Q = Au$ adopts when the flow is critical. For a wide, rectangular channel ($A = Wh$ and $\bar{h} = h$), it is

$$h_c \;=\; \left(\frac{Q^2}{gW^2}\right)^{\frac{1}{3}} . \tag{15.27}$$

A way of determining whether a flow is subcritical or supercritical is to compare its actual depth h to the critical depth h_c for that flowrate: If $h > h_c$, the flow is subcritical, while for $h < h_c$ the flow is supercritical.

15.4 Gradually Varied Flow

We now turn our attention to slowly varying flows, in which downstream variations play a role ($\partial/\partial x \neq 0$), but continue to restrict our attention to steady flows ($\partial/\partial t = 0$). With the velocity obtained in terms of the water depth and discharge [$u = Q/A$ according to (15.22)], the momentum equation (15.10) becomes:

$$-\,\frac{Q^2}{A^3}\,\frac{dA}{dh}\,\frac{dh}{dx} \;+\; g\,\frac{dh}{dx} \;=\; gS \;-\; C_D\,\frac{Q^2}{R_h A^2} \;,$$

which can be simplified by noting as earlier that dA/dh is the width W at the surface (see Figure 15.5). Furthermore, with $A/W = \bar{h}$, the average depth, it becomes:

$$-\,\frac{Q^2}{A^2\bar{h}}\,\frac{dh}{dx} \;+\; g\,\frac{dh}{dx} \;=\; gS \;-\; C_D\,\frac{Q^2}{R_h A^2} \;.$$

Now, gathering the dh/dx terms together and dividing by g, we obtain:

$$\left(1 \;-\; \frac{Q^2}{gA^2\bar{h}}\right)\frac{dh}{dx} \;=\; S \;-\; \frac{C_D Q^2}{gR_h A^2} \;. \tag{15.28}$$

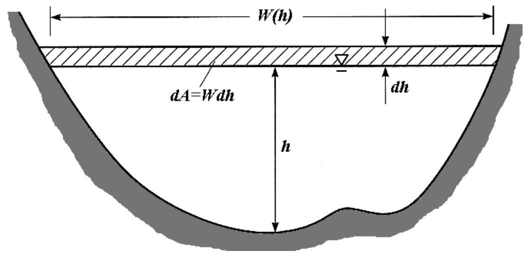

Figure 15.5: Relation between cross-sectional area A, width W and water depth h in a non-rectangular channel. When the water depth rises incrementally by dh, the area increases by Wdh.

The fraction inside the parentheses on the left-hand side is equal to $u^2/g\bar{h}$ and is thus the Froude number squared. The equation governing the downstream variation of water depth then takes the form:

$$\left(1 - Fr^2\right) \frac{dh}{dx} = S \left(1 - \frac{C_D Q^2}{g S R_h A^2}\right). \tag{15.29}$$

As we can note, it is not a priori clear whether the depth increases ($dh/dx > 0$) or decreases ($dh/dx < 0$) in the downstream direction, because each parenthetical expression can be either positive or negative. A number of possibilities arise, which we shall discuss them in the simpler case of a wide rectangular channel.

Wide rectangular channel

When the river channel is wide and rectangular, the average depth $\bar{h}$ and maximum depth h are the same, the cross-sectional area A is Wh, the wetted perimeter P is nearly the width W, making the hydraulic radius $R_h = A/P$ nearly equal to the water depth. With these simplifications, the preceding equation (15.29) becomes:

$$\left(1 - \frac{Q^2}{gW^2h^3}\right) \frac{dh}{dx} = S \left(1 - \frac{C_D Q^2}{gSW^2h^3}\right),$$

or

$$\left(1 - \frac{h_c^3}{h^3}\right) \frac{dh}{dx} = S \left(1 - \frac{h_n^3}{h^3}\right), \tag{15.30}$$

where h_c and h_n are respectively the critical and normal depths [see (15.27) and (15.25)]:

$$h_c = \left(\frac{Q^2}{gW^2}\right)^{\frac{1}{3}} \quad \text{and} \quad h_n = \left(\frac{C_D Q^2}{gSW^2}\right)^{\frac{1}{3}}. \tag{15.31}$$

Because the drag coefficient C_D depends, although weakly, on the water depth h, according to (15.19) or some other formula proposed by various authors, the dependence of right-hand side of (15.30) on the variable h is more complicated than it appears. Nonetheless, the sign of the right-hand side is determined by comparing the

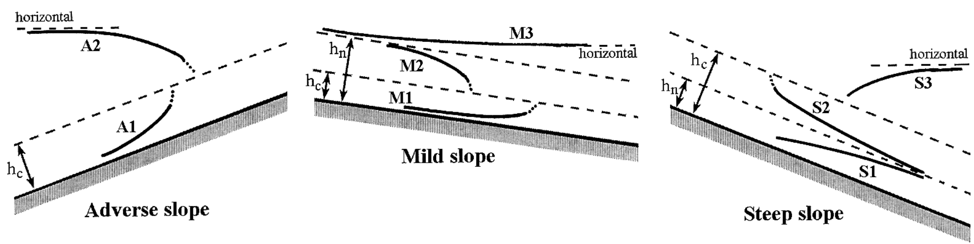

Figure 15.6: The profiles of the water surface in the eight possible cases of gradually varied flow.

actual water depth h value with the normal depth h_n, whatever its exact value may be.

We also note that, because h_n depends on h (via C_D) but h_c does not, there cannot exist a situation in which h_n and h_c are equal to each other over any finite distance x. Indeed, if such were the case, Equation (15.30) would reduce to $dh/dx = S$, yielding $h(x) = Sx$+constant, which makes h not constant and with it C_D and h_n. In other words, the case $h_n = h_c$ cannot arise and needs not be considered.

Various possible cases

Equation (15.30) can be cast as

$$\frac{dh}{dx} = S \, \frac{h^3 - h_n^3}{h^3 - h_c^3} \, , \tag{15.32}$$

which shows that the sign of dh/dx depends on how the actual water depth h compares to both the critical and normal depths, h_c and h_n.

The bottom slope S is usually positive as rivers flow downhill. However, there are cases when the slope may be locally negative, forcing the water to flow over a rising bottom. A prime example is the overflow from a lake, in which the water flows from a deeper basin over a sill and then down along a river channel. In the case of an *adverse slope*, the parameter S is negative and with it the normal depth h_n. With this in mind, the following cases can arise: The normal depth h_n can be (1) negative, (2) positive and less than the critical depth h_c, or (3) positive and greater than h_c. For h_n to exceed h_c, the channel slope S must be sufficiently weak, namely

$$S \; < \; C_D. \tag{15.33}$$

In such case, the slope is said to be *mild*. In the contrary case, when (15.33) is not satisfied, the slope is said to be *steep*, and h_n falls below h_c.

And, for every one of these cases, the actual water depth h may lie in any interval defined by 0, h_c and h_n, if the latter is positive. This leads to the following set of eight possible cases:

Adverse slope ($S < 0$, $h_n < 0 < h_c$):
- A1: $0 < h < h_c$
- A2: $h_c < h$

Mild slope ($S > 0$, $0 < h_c < h_n$):
- M1: $0 < h < h_c$
- M2: $h_c < h < h_n$
- M3: $h_n < h$

Steep slope ($S > 0$, $0 < h_n < h_c$):
- S1: $0 < h < h_n$
- S2: $h_n < h < h_c$
- S3: $h_c < h$

It is straightforward to note that dh/dx is positive in the following 5 cases: A1, M1, M3, S1, S3, and negative in the 3 others: A2, M2, S2. Next, we note that when h approaches h_n it does so asymptotically (cases S1 and S2), but when it approaches h_c it does so in a singular way (dh/dx approaching infinity, cases A1, A2, M1 and M2). Finally, when h increases without bound (cases M3 and S3), Equation (15.32) yields $dh/dx \to S$, implying that the rise in water depth compensates for the drop in bottom, and the water surface becomes horizontal. The water profile in the eight cases is displayed in Figure 15.6.

The behavior of the water level in some of the cases displayed in Figure 15.6 appear to be quite odd at first glance, because they lead to a singularity ($dh/dx \to \infty$), but they make sense in combinations with one another, as shown in Figure 15.7.

15.5 Lake Discharge Problem

A practical problem in hydraulics is the determination of the discharge (volumetric flow rate) from a lake given its water level and the slope of the exit channel. Two cases are possible: Either the slope of the exit channel is mild or steep. The selection reduces to finding whether the channel slope S falls below or exceeds the value of the drag coefficient [see Inequality (15.33)] Since, the drag coefficient C_D in the exit channel is dependent on the water depth h and since the latter is not known until the discharge is determined, the solution must proceed by trial and errors. But, let us assume here that we have made a reasonable guess about the value of C_D and that we therefore know whether the slope of the exit channel is mild or steep.

The easier of the two cases is that of an exit channel with a steep slope ($S > C_D$). The flow in the stream draining the lake is then of type S1, S2 or S3. By virtue of Figure 15.6, it is clear that we can reject S3, because the lake is upstream, not downstream. The flow in the stream is therefore supercritical. And, of S1 and S2, it is quite clear that we need to select S2 because the flow goes from being deeper in the lake to shallower in the stream. The water velocity is virtually nil in the deep lake, and the Froude number there is nearly zero. Thus, the lake flow approaching the exit

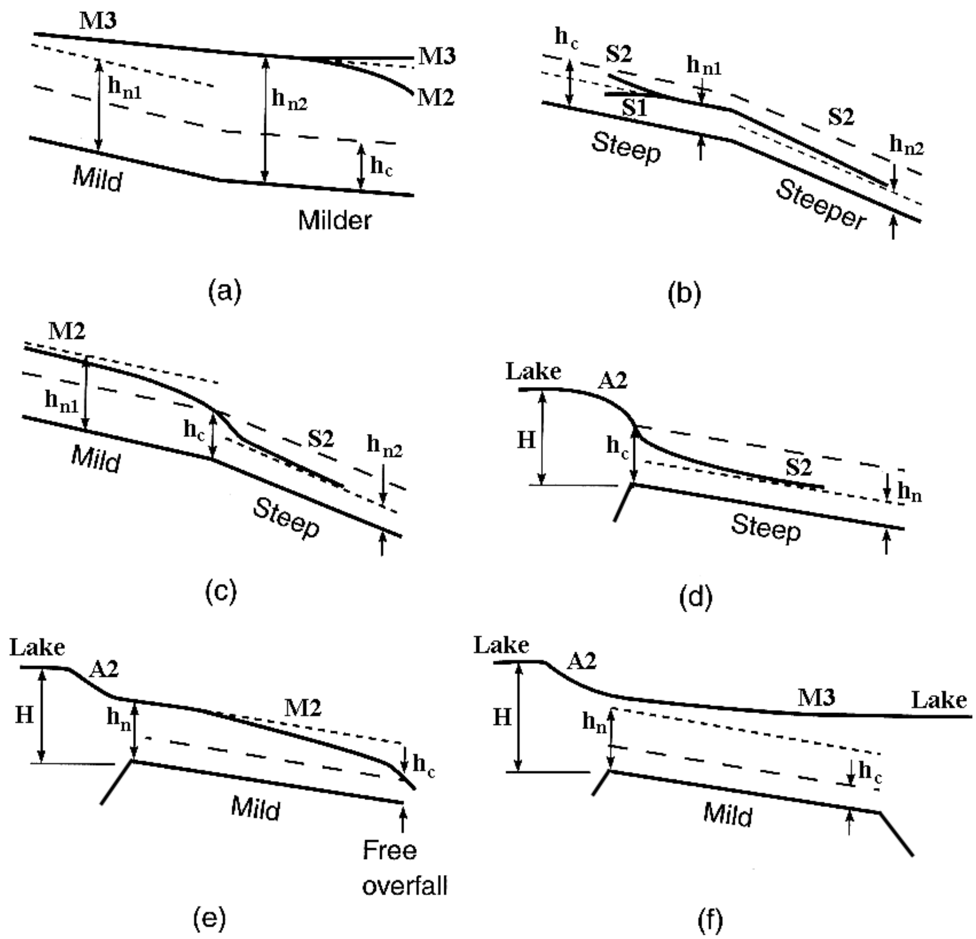

Figure 15.7: Combinations of gradually-varied flows: (a) stream passing from a mild to a milder slope; (b) change from a steep to a steeper slope; (c) change from a mild to a steep slope; (d) lake discharging in a river with steep slope; (e) lake discharging in a river with mild slope, which becomes steep further downstream; (f) lake discharging in a river with mild slope, which runs into another lake. [Adapted from Sturm, 2001]

is subcritical. As the flow needs to pass from subcritical in the lake to supercritical in the stream, it crosses criticality ($h = h_c$) at the transition from the lake to the stream, that is, at the sill point (highest bottom point), as indicated in Figure 15.7d, and the sill exerts control.

Generally, the bottom rises abruptly in the lake in the vicinity of the sill point, and we can consider the portion of the flow on the lake side of the sill as rapidly varied (frictionless). The Bernoulli principle holds, telling us that the sum $u^2/2 + g(h + b)$ is constant from the deep lake to the sill. On the deep side, $u \simeq 0$ whereas $h + b$ is equal to H, the elevation of the water surface in the lake above the height of the sill (see Figure 15.7d, with the datum taken as the sill level). At the sill, the velocity is critical, $u = \sqrt{gh}$ ($Fr = 1$), whereas b is zero. Conservation of the Bernoulli function then provides:

$$\frac{0}{2} + gH = \frac{gh}{2} + gh,$$

which yields

$$h = \frac{2}{3} H, \tag{15.34}$$

at the sill and, in turn,

$$u = \sqrt{gh} = \sqrt{\frac{2}{3} gH} , \tag{15.35}$$

at the sill, too. The discharge Q is the product Whu, where W is the channel width. The answer to the problem is thus:

$$Q = Whu = \left(\frac{2}{3}\right)^{3/2} WH\sqrt{gH} = 0.544 \, WH\sqrt{gH} . \tag{15.36}$$

The case of an exit channel with mild slope ($S < C_D$) is somewhat more complicated. Rejecting immediately type M1, because the lake is upstream and not downstream, we are left with a choice between flow types M2 and M3, each with control at the downstream end of the channel, that is, away from the lake. If we can assume that the channel is relatively long, then the flow at the head of the channel is the upstream asymptotic behavior of either M2 or M3, (as depicted in Figure 15.7e and 15.7f). Thus, we are brought to conclude that the water depth reaches the normal value ($h = h_n$) at the head of the channel. Assuming as in the case of a steep slope that the lake flow approaching the sill varies rapidly from rest, we can again apply the Bernoulli principle and write:

$$\frac{0}{2} + gH = \frac{u^2}{2} + gh_n,$$

which yields the velocity at the sill point:

$$u = \sqrt{2g(H - h_n)} . \tag{15.37}$$

In terms of the discharge Q, the normal water depth is $h_n = (C_D Q^2/gSW^2)^{1/3}$ and the corresponding water velocity is $u = Q/Wh_n = (gSQ/C_D W)^{1/3}$. Using these in (15.37) yields the value of Q:

$$Q \ = \ \left[\frac{1}{2} \left(\frac{S}{C_D} \right)^{2/3} \ + \ \left(\frac{C_D}{S} \right)^{1/3} \right]^{-3/2} WH\sqrt{gH} \ , \tag{15.38}$$

which reverts to (15.36) when $S = C_D$, providing continuity between the two cases of mild and steep slopes.

If the channel draining the lake does not preserve its mild slope for a long distance, then one needs to start from the end point of this channel stretch where control is occurs and integrate in the upstream direction all the way to the sill. This is quite complicated because numerical integration requires a value for Q, which is not yet known. One has therefore to proceed by successive trials until $u^2/2 + gh$ equals gH at the sill, to establish connection with the lake.

15.6 Rapidly Varied Flow

Bernoulli principle

In considering rapidly varied flow, friction may be neglected and the drag coefficient C_D is set to zero. Equation (15.10) in steady state reduces to:

$$u \, \frac{du}{dx} \ + \ g \, \frac{dh}{dx} \ + \ g \, \frac{db}{dx} \ = \ 0,$$

in which we have introduced the elevation $b(x)$ of the channel bottom, so that the slope is minus its gradient: $S = -db/dx$.

This can be integrated over distance to obtain the Bernoulli principle:

$$\frac{u^2}{2} \ + \ gh \ + \ gb \ = \ B \ = \ \text{a constant.} \tag{15.39}$$

Elimination of the velocity u by virtue of conservation equation (15.22) of the flow rate Q yields

$$\frac{Q^2}{2A^2} \ + \ gh \ + \ gb \ = \ B. \tag{15.40}$$

This constitutes an algebraic equation for the water depth h since the cross-sectional area A is a function of h. Given a cross-sectional profile $A(h)$ and bottom elevation b, we can in principle solve the equation for h and, as either or both of these two properties change in the downstream direction, so does the value of h.

For a channel of rectangular cross-section, $A = Wh$ with W the channel width, Equation (15.40) becomes:

Figure 15.8: Water flowing down a weir, which is a type of rapidly varied flow. The Bernoulli principle may be applied between Points **1** and **2**. [Photo ©Chanson 2000]

$$\frac{Q^2}{2W^2h^2} + gh + gb = B. \tag{15.41}$$

The parameters are the discharge Q, the channel width W, the bottom elevation b, and the level B of energy in the flow. For given values of these parameters, the water depth h can be calculated. The parameters Q and B are constants along the stream, but rapid changes can occur in the width W and bottom elevation b. The water depth h then adapts locally, and this is the essence of a rapidly varied flow.

Note that Equation (15.41) can be turned into a cubic polynomial, which may have one, two or three real roots. Whether any, some or all of the real roots are positive, which is required for h to be physically realizable, needs to be investigated.

It is the tradition in civil engineering, of which hydraulics is a discipline, to use variables that correspond to vertical distances, called *heads*. To this effect, Equations (15.39) and (15.41) are divided by g:

$$\frac{u^2}{2g} + h + b = \frac{B}{g} \tag{15.42}$$

$$\frac{Q^2}{2gW^2h^2} + h + b = \frac{B}{g}. \tag{15.43}$$

The term $u^2/2g = Q^2/2gW^2h^2$ is called the *velocity head*, whereas h and b are obviously vertical heights, that of water above the bottom of the channel and that of the bottom of the channel above sea level, respectively.

Specific energy

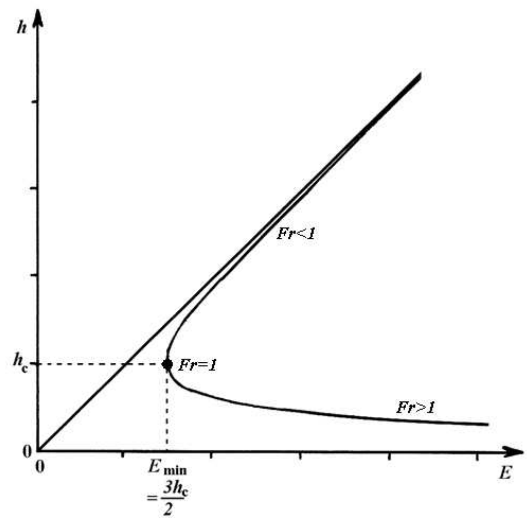

Figure 15.9: Variation of specific energy E with water depth. It is customary in this type of plot to have h along the vertical axis because it corresponds to a vertical length.

If we consider two points along the flow (Figure 15.8), with Point **1** upstream of Point **2**, Equation (15.43) requires

$$\frac{Q^2}{2gW_1^2h_1^2} + h_1 = \frac{Q^2}{2gW_2^2h_2^2} + h_2 + \Delta b,$$

in which $\Delta b = b_2 - b_1$ is the change in bottom elevation. From this expression, it is clear that the sum of the velocity head and water depth must change if there is any change in channel elevation. In consequence, it is instructive to consider this sum of two terms, which was first introduced by Bakhmeteff (1932) and has come to be called the *specific energy*. By definition therefore,

$$E = \frac{Q^2}{2gW^2h^2} + h \tag{15.44}$$

for a channel with discharge Q and rectangular cross-section of width W.

We note the following interchange in E upon varying h: When one of the two terms increases, the other necessarily decreases. The limits of h going to zero and to infinity are both $E \to \infty$, and since E is finite for finite values of h and is obviously well behaved, it follows that E reaches a minimum for some value of h (Figure 15.9). Setting to zero the derivative with respect to h, we obtain:

$$-\frac{Q^2}{gW^2h^3} + 1 = 0,$$

of which the solution is

$$h_c = \left(\frac{Q^2}{gW^2}\right)^{\frac{1}{3}}, \tag{15.45}$$

in which we recognized the *critical depth* [see (15.27). The minimum value of E is obtained by setting $h = h_c$:

$$E_{\min} \; = \; \frac{3}{2} \left(\frac{Q^2}{gW^2} \right)^{\frac{1}{3}} \; = \; \frac{3h_c}{2} \; . \tag{15.46}$$

A plot of the function $E(h)$ is provided in Figure 15.9. It is clear that there exist two possible values for h when E exceeds its minimum, one when it is at its mimimum, and none when it falls below its minimum. [There also exists another real h root but it is always negative and not included in the plot.] When the channel width W increases, both $E_{\min}$ and h_c decrease, and the curve moves inward, being squeezed inside the wedge defined by the horizontal axis and the bissectrix ($h = E$ line), and when W decreases the curve moves away from the apex of the wedge.

The Froude number is

$$Fr \; = \; \frac{u}{\sqrt{gh}} \; = \; \sqrt{\frac{Q^2}{gW^2h^3}} \; = \; \left(\frac{h_c}{h} \right)^{\frac{3}{2}} . \tag{15.47}$$

Therefore, the upper branch of E corresponds to *subcritical flow* ($Fr < 1$) and the lower branch to *supercritical flow* ($Fr > 1$). The minimum of E corresponds to a critical state ($Fr = 1$), and this is why h_c is called the critical depth. For any value higher than $E_{\min}$, there thus exist two states, one subcritical (thick and slow, called *fluvial*) and one supercritical (thin and fast, called *torrential*), but no state is realizable when E falls below $E_{\min}$.

Flow over a bump

Let us now consider what happens when the stream encounters a bump along the bottom, while its width remains unchanged. So, b is now a function of x which increases and then decreases. Equations (15.43) requires the specific energy to change according to

$$E \; + \; b \; = \; \frac{B}{g} \; , \tag{15.48}$$

in which B/g is a constant. Thus, E must decrease where b increases and vice versa. At the top of the bump, E is lower than what it was upstream by exactly the height of the bump, say Δb. If the bump is modest and the flow is subcritical upstream, the point on the specific-energy diagram (Figure 15.10a) slides downward, the flow becomes thinner and a bit faster. Physically, the flow is constricted from below and must accelerate to accommodate an unchanged flowrate. But, acceleration demands a force, and the surface must fall somewhat so that the flow can slide downwards to accelerate. Another way of understanding this is to realize that the increase in kinetic energy necessary to squeeze the flow above the bump can only be at the expense of potential energy, and, as the potential energy falls, so does the surface.

Note the positive feedback in the situation. The flow is squeezed from below by the bump and must simultaneously experience a squeeze from the top. This can obviously

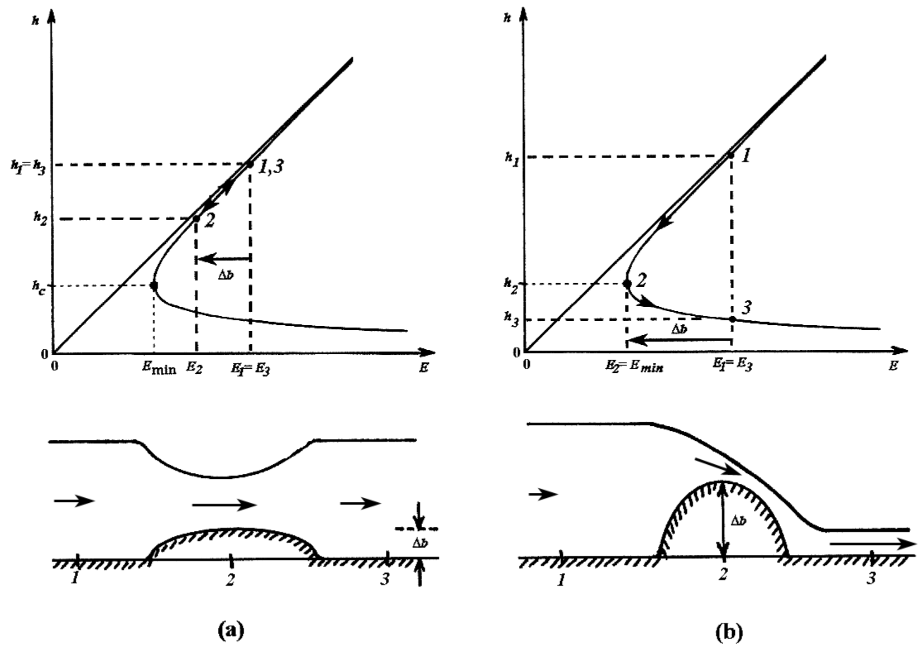

Figure 15.10: The possible ways by which a free-surface flow adapts to a bottom bump: (a) The bump is modest, and the flow remains subcritical all along, exhibiting only a dip over the bump; (b) the bump is high, and the flow becomes critical at the top of the bump and supercritical thereafter.

lead to a problem: If the bump is of sufficient height, the required drop in the surface level might become excessive and intersect the raised bottom. This is what happens when the height of the bump exceeds the difference between the upstream value of E and its minimum E_{min}. The value of E should fall below its minimum but this may not happen. The flow cannot pass over the bump, at least not all of it. The flow is said to be *choked*.

The situation becomes unsteady. Water arrives at the obstacle faster than it can pass over it and accumulates. This accumulation in turn raises the water level upstream, and, with it, the potential energy of the flow. Mathematically, the value of the Bernoulli function B is now augmenting, and with it the specific energy E of the flow upstream. This will continue until E has been raised just enough that the new difference $E - E_{\text{min}}$ above the minimum can match the height Δb of the bump.

On the specific-energy diagram, the point slides down along the upper branch on the climbing side of the bump and reaches the minimum at the top (Figure 15.10b). Downstream, the point does not proceed reversibly but keeps on sliding along the

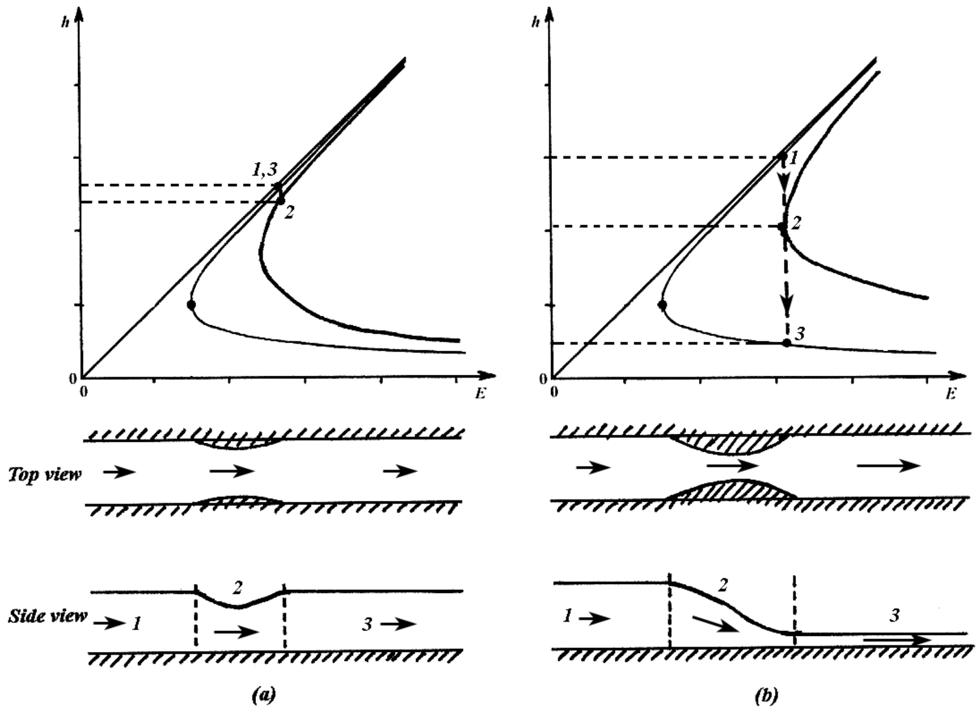

Figure 15.11: The possible ways by which a free-surface flow passes through a narrowing section: (a) The narrowing is modest, and the flow remains subcritical all along, exhibiting only a dip in the constriction; (b) the narrowing is significant, and the flow becomes critical at the narrowest section and supercritical thereafter.

lower branch of the curve. The flow has become thin and fast, that is, supercritical. The bump acts as a dam and spillway (see Figure 15.8 for example).

Both previous cases assumed that the oncoming flow was subcritical. Should it be supercritical, again one of two things can happen. If the height of the bump is modest, the flow slows down and thickens over the bump, because on the specific energy diagram the point rises as it moves to the left and then relaxes. If the height of the bump is large, the flow switches from being supercritical to subcritical. As it will be shown later, however, supercritical flows are unstable and do not persist. An approaching supercritical flow is therefore quite unlikely.

Flow in a narrowing channel

Another way by which a flow can be choked and pass from subcritical to supercritical state is through a narrowing of the channel width. Now, b remains constant but the width W of the channel experiences a local decrease, say from W upstream and downstream to $W_{\min}$ at the narrowest point (Figure 15.11). Since the bottom elevation b remains unchanged, the specific energy, too, remains constant by virtue of (15.48), but as the width W decreases, the specific-energy minimum (15.46) increases and the curve in the E–h diagram moves to the right.

If the narrowing of the channel is modest (Figure 15.11a), the water depth drops some as the channel narrows and recovers if the channel widens afterwards. But, if the restriction in channel cross-section is significant, then the flow undergoes a transition from subcritical to supercritical state (assuming that it was subcritical upstream), with the critical point occurring at the narrowest cross-section (see Figure 15.11b).

Flooding

An interesting situation occurs when a river overflows its normal channel and spills onto a broader floodplain. The floodplain naturally forms a new and wider channel but the compounded cross-section of natural channel plus floodplain may no longer be idealized as a channel of rectangular cross-section.

Consider a channel of arbitrary cross-section, for which the cross-sectional area $A(h)$ is some complicated function of the water depth h. The specific energy depends on h in the following general way:

$$E \;=\; \frac{Q^2}{2gA^2(h)} \;+\; h, \qquad (15.49)$$

and reaches an extremum with respect to h when its derivative dE/dh vanishes, which occurs when:

$$\frac{1}{A^3(h)}\,\frac{dA}{dh} \;=\; \frac{g}{Q^2}\;.$$

As Figure 15.5 shows, the channel width at the surface is such that $dA = W dh$ and thus $dA/dh = W$. The preceding equation can therefore be expressed as:

$$\frac{A^3(h)}{W(h)} \;=\; \frac{Q^2}{g}\;, \qquad (15.50)$$

which by virtue of its nonlinearity may admit more than one solution. If there is a unique solution, it must be a minimum since E tends to positive infinity when h goes to zero and to infinity. If (15.50) admits more than one root, then it is most likely that the solutions come in a set of 3, 5 etc. (barring the exceptional cases of double roots). With three solutions, we expect two minima separated by one maximum.

Defining the averaged depth $\bar{h}$ as the cross-sectional area divided by the surface width, namely

$$\bar{h} \;=\; \frac{A}{W}\;, \qquad (15.51)$$

then (15.50) can be recast as

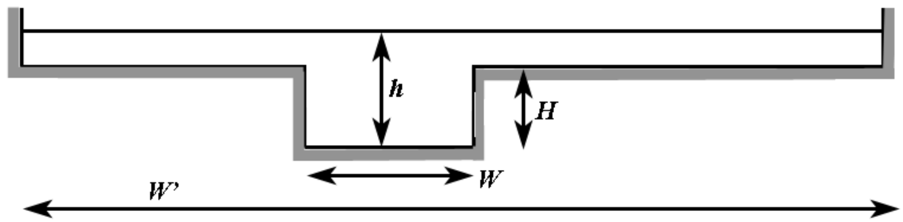

Figure 15.12: An idealized river channel and its floodplain. The river channel has a width W and depth H, while the floodplain has a width W' significantly larger than W.

$$A^2\bar{h} \;=\; \frac{Q^2}{g}\;. \tag{15.52}$$

If the Froude number is defined in terms of the averaged depth as

$$Fr \;=\; \frac{u}{\sqrt{g\bar{h}}}\;, \tag{15.53}$$

then, using $u = Q/A$, we have

$$Fr \;=\; \sqrt{\frac{Q^2}{gA^2\bar{h}}}\;, \tag{15.54}$$

which reaches 1 whenever E reaches an extremum, as for the rectangular cross-section.

Channel with floodplain

A channel and its surrounding floodplain may be idealized as a wide and shallow rectangular channel with a deeper and narrower rectangular channel embedded within it, as depicted in Figure 15.12.

In the non-flooding case $h \le H$, the cross-sectional area is $A = Wh$, the velocity $u = Q/Wh$, and the specific energy

$$E \;=\; \frac{Q^2}{2gW^2h^2} \;+\; h,$$

whereas in the flooding case $h > H$, the cross-sectional area is $A = WH + W'(h-H) = W'h - (W'-W)H$, the velocity $u = Q/[W'h - (W'-W)H]$ and the specific energy

$$E \;=\; \frac{Q^2}{2g[W'h-(W'-W)H]^2} \;+\; h.$$

Extrema of E occur for

$$h = \left(\frac{Q^2}{gW^2}\right)^{\frac{1}{3}} \qquad \text{for} \quad h \leq H$$

$$h = \left(\frac{Q^2}{gW'^2}\right)^{\frac{1}{3}} + \frac{W' - W}{W'}\, H \qquad \text{for} \quad h > H,$$

and these are distinct roots if

$$\left(\frac{Q^2}{gW^2}\right)^{\frac{1}{3}} < H < \left(\frac{Q^2}{gW'^2}\right)^{\frac{1}{3}} + \frac{W' - W}{W'}\, H \tag{15.55}$$

These inequalities can be recast as:

$$\frac{W}{W'} < \frac{Q^2}{gW^2H^3} < 1 \tag{15.56}$$

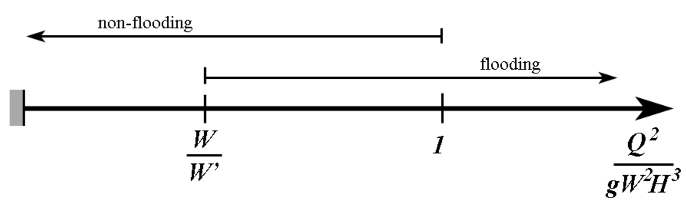

Figure 15.13: Existence of minima of the specific energy depending on the river discharge Q in the case of a channel plus floodplain. In the intermediate range, there are two roots, one corresponding to flooding and the other to the river confined to its bed.

Figure 15.13 depicts the situation in terms of the normalized discharge Q^2/gW^2H^3, and we note that for low discharges ($Q^2/gW^2H^3 < W/W'$), there is a single minimum to E, which is the one of the rectangular river channel. The possibilities are fast, supercritical flow confined to the river bed but slow, subcritical flow possibly flooding.

For a large discharge ($Q^2/gW^2H^3 > 1$), there is also a single minimum for E, but this one corresponds to a critical water depth in the flooding configuration ($h > H$). Slow, subcritical flow must be flooding, whereas fast supercritical flow may or may not be contained in the river bed.

The intricate case is the intermediate one ($W'/W < Q^2/gW^2H^3 < 1$), when both inequalities can be met simultaneously. The specific energy in this case exhibits two minima, one for $h < H$ and the other for $h > H$, separated by a maximum at $h = H$, where the E-curve has a discontinuous derivative (Figure 15.14). There can be up to four different flow states corresponding to the same discharge Q and specific energy E, two subcritical and two supercritical states. The elucidation of the various cases is tedious, and we shall leave the topic by simply remarking that the prediction of

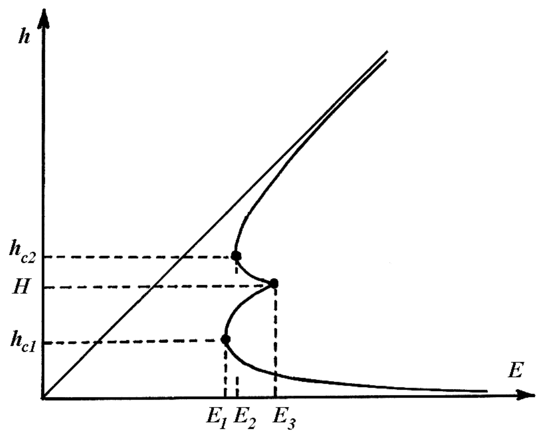

Figure 15.14: The specific-energy diagram in the case of a compound channel (river bed plus floodplain) when the discharge is intermediate ($W/W' < Q^2/gW^2H^3 < 1$). The specific energy E exhibits two minima and one local maximum.

flooding is not as straightforward as it may first appear. In particular, it is not clear whether placing sand bags to avoid flooding at one location (in an inhabited section of the river, for example) may or may not choke the flow. Should it choke the flow, it is likely to cause more flooding upstream.

15.7 Hydraulic Jump

Supercritical flows are unstable and, under slight perturbations, which are always unavoidable, naturally reverse to subcritical conditions. The transition from an upstream, thin and fast flow to a downstream, thick and slow flow is called a *hydraulic jump*. The jump appears as a retrogressive wave that tries to creep upstream but cannot because the flow opposes its progression. There is a loss of energy in the hydraulic jump, i.e., the specific energy E [see (15.44)] drops across the jump. Thus, the Bernoulli principle is invalidated, but both mass conservation and momentum budget continue to hold and can be used to determine the changes in the flow across the jump.

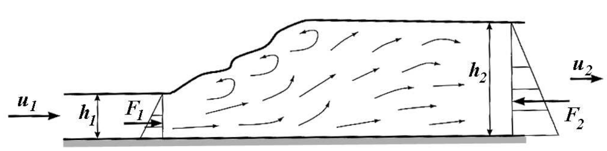

Figure 15.15: A hydraulic jump and the attending notation.

In a channel of uniform width (W = constant) and with flat bottom ($b = 0$), we may write (Figure 15.15):

$$\text{Mass conservation:} \qquad \rho h_1 u_1 = \rho h_2 u_2, \tag{15.57}$$

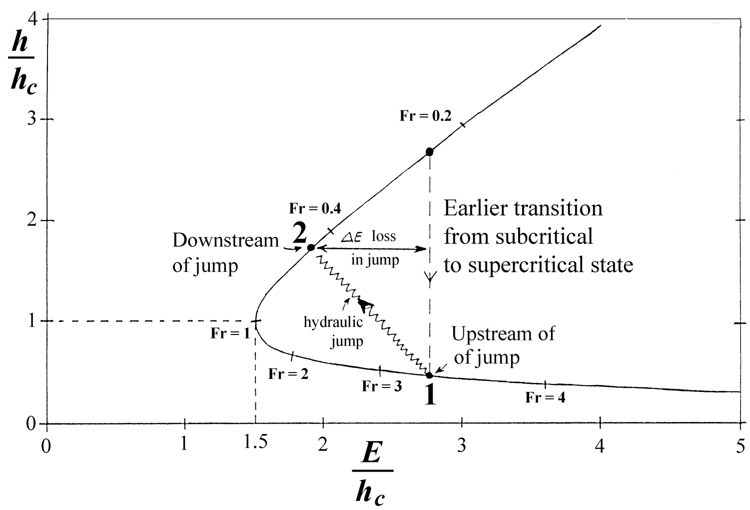

Figure 15.16: Change in flow characteristics caused by a hydraulic jump.

$$\text{Momentum budget:} \qquad \rho h_1 u_1^2 \; + \; \frac{\rho g h_1^2}{2} \; = \; \rho h_2 u_2^2 \; + \; \frac{\rho g h_2^2}{2}, \tag{15.58}$$

where subscripts 1 and 2 refer respectively to the upstream and downstream conditions. The momentum budget is quite simple because on a flat horizontal bottom, there is no force on the flow besides the hydrostatic pressure force. (Recall that bottom friction is neglected in rapidly varied flows, which is the case in a hydraulic jump.)

Combining the preceding two equations, we obtain after some algebra the ratios of the downstream velocity and height to their respective upstream values:

$$\frac{u_1}{u_2} \; = \; \frac{h_2}{h_1} \; = \; \left(\frac{Fr_1}{Fr_2}\right)^{\frac{2}{3}} \; = \; \frac{\sqrt{1 + 8Fr_1^2} - 1}{2} \, , \tag{15.59}$$

where $Fr_i = u_i/\sqrt{gh_i}$ is the Froude number at position i. Equation (15.59) shows that the amount of change in the jump is determined solely by the upstream Froude number. Thus, given upstream conditions h_1 and u_1, it is a simple matter to predict the downstream characteristics h_2 and u_2 of the flow, and, from them, to calculate the energy loss across the jump, called the *head loss.* The head loss across a hydraulic jump is equal to:

$$\Delta E \; = \; E_1 - E_2 \; = \; \frac{(h_2 - h_1)^3}{4h_1 h_2} \, , \tag{15.60}$$

which is positive as long as h_2 exceeds h_1, that is, if the flow switches from supercritical to subcritical state. This energy loss can be computed from the upstream Froude number and used to locate the post-jump point on the specific energy diagram, as done in Figure 15.16.

Consider now the case of a river in which the flow is repeatedly made supercritical by multiple dams and lateral constrictions, and each time reverts to a subcritical state by means of hydraulic jumps. With every successive jump, the flow loses energy, until there is no more energy to lose. This occurs when the point on the specific-energy diagram has migrated to $E = E_{\min}$, at which point $h = h_c$. Thus, on a bottom that is horizontal (except for the occasional dams), the critical state is the attractor. This also holds true for the case of a mild slope. As seen in Figure 15.6, the normal flow is the attractor only on a steep slope.

15.8 Air-Water Exchanges

Surface chemistry

Wherever water is in contact with air, such as in rivers, ponds, lakes and oceans, a chemical transfer occurs between the two fluids. Some of the water evaporates creating moisture in the atmosphere while some of the air dissolves into the water. Different constituents of air (N_2, O_2, CO_2 etc.) dissolve to different degrees and in amounts that depend on temperature.

At equilibrium, a relation known as Henry's Law exists between the amounts of the gas dissolved in the water and the amount present in the atmosphere:

$$[\text{gas}]_{\text{in water}} \;=\; K_H \; p_{\text{gas in air}} \tag{15.61}$$

which states a proportionality between the concentration of gas dissolved in the water ($[\text{gas}]_{\text{in water}}$, in moles per liter, noted M), and the partial pressure[6] of the same gas in the air ($p_{\text{gas in air}}$, in atmosphere, noted atm). The coefficient of proportionality is the so-called Henry's Law constant, K_H (in M/atm). Table 15.2 lists its values for oxygen and carbon dioxide at various temperatures.

Example 16.1

Let us apply Henry's Law to dissolved oxygen (DO) in water at two different temperatures. At 15°C, Table 15.2 provides K_H = 0.0015236 M/atm, which yields under a standard partial pressure of oxygen in the atmosphere equal to 0.2095 atm:

$$[O_2] \;=\; (0.0015236 \text{ M/atm}) \;\times\; (0.2095 \text{ atm}) \;=\; 3.19\ 10^{-4}\ M.$$

[6]The partial pressure of a gas species in a gas mixture is the pressure times the mole fraction of that species in the mixture. For example, oxygen is 20.95% of the air on a molar basis and, therefore, P_{O_2} is 20.95% of the atmospheric pressure, or 0.2095 atm under standard conditions.

Table 15.2: Values of Henry's Law constant for oxygen and carbon dioxide.

Temperature (°C)	Oxygen (M/atm)	Carbon dioxide (M/atm)
0	0.0021812	0.076425
5	0.0019126	0.063532
10	0.0016963	0.053270
15	0.0015236	0.045463
20	0.0013840	0.039172
25	0.0012630	0.033363

And, since the molecular weight of the oxygen molecule is $2 \times 16 = 32$ g/mole = 32000 mg/mole, we deduce

$$\mathrm{DO} \ = \ 32000 \ \mathrm{mg/mole} \ \times \ 3.19 \ 10^{-4} \ \mathrm{moles/L} \ = \ 10.21 \ \mathrm{mg/L}.$$

Likewise, at 20°C: K_H is 0.0013840 M/atm, leading successively to $[O_2] = 0.0013840 \times 0.2095 = 2.89 \ 10^{-4}$ M and DO $= 32000 \times 2.89 \ 10^{-4} = 9.23$ mg/L.

Dissolved-oxygen values determined from Henry's Law are realized only when an equilibrium is reached between the water and air, which is not always the case. Thus, a distinction must be made between this equilibrium value, called the *saturated value* denoted DO_s, and the actual value, DO. Table 15.3 recapitulates the saturated values of dissolved oxygen for various temperatures and under a standard atmospheric pressure.

Reaeration and volatilization

Henry's Law expresses an equilibrium between air and water, but not all situations are at equilibrium because processes in one medium may skew the situation. An example is the consumption of dissolved oxygen by bacteria in dirty water. The oxygen depletion disrupts the surface equilibrium, and the resulting imbalance draws a flux of new oxygen from the air into the water. In other words, equilibrium corresponds to a state of no net flux between the two fluids, whereas displacement away from equilibrium is characterized by a flux in the direction of restoring the situation toward equilibrium.

A useful way of determining the flux in non-equilibrium situation is the so-called *thin-film model*. According to this model, both fluids have thin boundary layers (a few micrometers thick), in which the concentration of the substance under consideration varies from the value inside that fluid rapidly but continually to a value at the interface between the two fluids, as depicted in Figure 15.17.

Table 15.3: Values of saturated dissolved oxygen DO_s as function of temperature, in pure freshwater under standard atmospheric pressure.

Temperature (°C)	Oxygen (mg/L)	Temperature (°C)	Oxygen (mg/L)
0	14.6	13	10.6
1	14.2	14	10.4
2	13.8	15	10.2
3	13.5	16	10.0
4	13.1	17	9.7
5	12.8	18	9.5
6	12.5	19	9.4
7	12.2	20	9.2
8	11.9	21	9.0
9	11.6	22	8.8
10	11.3	23	8.7
11	11.1	24	8.5
12	10.8	25	8.4

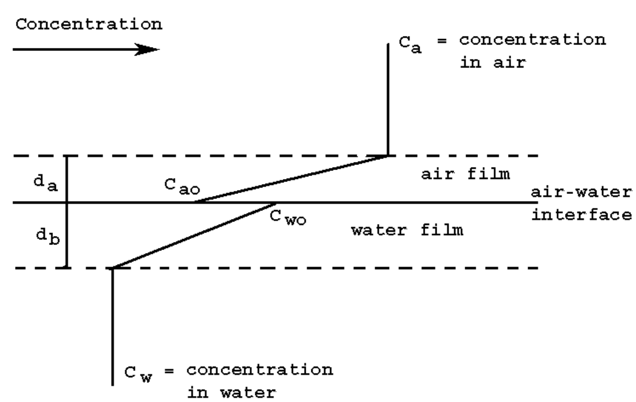

Figure 15.17: The thin-film model at the air-water interface during a situation away from equilibrium.

If we denote by C_a and C_w the air and water concentrations of the substance away from the interface, by C_{ao} and C_{wo} the concentrations at the interface, by d_a and d_w the thicknesses of the film layers, and by D_a and D_w the air and water molecular diffusivities, we can write two statements. First, because there is no accumulation or depletion of the substance at the interface itself, the diffusion flux in the air must exactly match that in the water:

$$q \;=\; D_a \,\frac{C_{ao} - C_a}{d_a} \;=\; D_w \,\frac{C_w - C_{wo}}{d_w} \tag{15.62}$$

Also, instantaneous equilibrium may be assumed at the level of the interface:

$$C_{wo} \;=\; K_H \; p_o \;=\; K_H \; RT \; C_{ao}, \tag{15.63}$$

where p_o is the partial pressure of the substance in the air at the level of the interface, equal to $RT\;C_{ao}$ according to the ideal-gas law. Replacing C_{wo} by this value in (15.62) and solving for C_{ao}, we obtain:

$$C_{ao} \;=\; \frac{d_w D_a C_a \;+\; d_a D_w C_w}{d_w D_a \;+\; d_a D_w K_H RT}\;, \tag{15.64}$$

and the flux q can be expressed as:

$$\begin{aligned} q \;&=\; \frac{D_a D_w}{d_w D_a \;+\; d_a D_w K_H RT}\,(C_w \;-\; K_H RT C_a) \\ &=\; \frac{1}{\frac{d_w}{D_w} \;+\; \frac{d_a}{D_a} K_H RT}\,(C_w \;-\; K_H p), \end{aligned} \tag{15.65}$$

where p is the partial pressure of the substance in the air away from the interface. The outcome is that the flux is proportional to the departure $(C_w \;-\; K_H p)$ from equilibrium. Lumping the front fraction as a single coefficient of reaeration k_r, we write:

$$q \;=\; k_r\;(C_w \;-\; K_H p). \tag{15.66}$$

Naturally, this flux is in the direction of restoration toward equilibrium. If the concentration in the water is less than at equilbrium ($C_w \;<\; K_H p$), then the flux is negative ($q \;<\; 0$), meaning downward from air into water, and vice versa if the concentration in the water exceeds that of equilibrium ($C_w \;>\; K_H p \;\rightarrow\; q \;>\; 0$), or upward from water into the air.

In the particular case of oxygen, equilibrium is achieved at saturation, when the actual dissolved oxygen DO equals the maximum, saturated amount DO_s. Thus, the reaeration flux is expressed in terms of the oxygen deficit:

$$k_r\;(\mathrm{DO}_s \;-\; \mathrm{DO}),$$

which is counted positive if the water is taking oxygen from the air. The preceding expression is on a per-area, per-time basis. To obtain the rate of oxygen intake, we multiply by the area A_s of the water surface exposed to the air:

Table 15.4: Typical values of the reaeration coefficient for various streams. [From Peavy, Rowe and Tchobanoglous, 1985]

Stream type	K_r at 20°C (in 1/day)
Sluggish river	0.23–0.35
Large river of low velocity	0.35–0.46
Large stream of normal velocity	0.46–0.49
Swift streams	0.69–1.15
Rapids and waterfalls	> 1.15

$$R \;=\; A_s k_r \;(\mathrm{DO}_s \;-\; \mathrm{DO}). \tag{15.67}$$

The coefficient of reaeration k_r depends on temperature. The formula most often used is

$$k_r(\text{at } T) \;=\; k_r(\text{at } 20°\text{C})\; 1.024^{T-20} \;, \tag{15.68}$$

where T is here the temperature in degrees Celsius. The value at the reference temperature of 20°C depends on the degree of agitation (turbulence) in the water, which in turns depends on the velocity and depth of the water. A useful empirical formula is

$$k_r(\text{at } 20°\text{C}) \;=\; 3.9 \left(\frac{u}{h}\right)^{1/2} \;, \tag{15.69}$$

In this formula, which is dimensionally inconsistent, the stream velocity u and depth h must be expressed in m/s and m, respectively, to obtain the k_r value in m/day. In most applications, the reaeration coefficient has to be divided by the water depth, and some authors define the ratio

$$K_r \;=\; \frac{k_r}{h} \tag{15.70}$$

as the reaeration coefficient. Table 15.4 lists typical values of this ratio.

15.9 Dissolved Oxygen

Biological oxygen demand

By far the most important characteristic determining the quality of a river or stream is its dissolved oxygen. While the saturated value DO_s is rarely achieved, a stream can nonetheless be considered healthy as long as its dissolved oxygen DO exceeds 5 mg/L. Below 5 mg/L, most fish, especially the more desirable species such as trout, do not survive. Actually, trout and salmon need at least 8 mg/L during their embryonic and larval stages and the first 30 days after hatching.

Except for pathogens, organic matter in water is generally not harmful in and of itself but may be considered as a pollutant because its bacterial decomposition generates a simultaneous oxygen depletion. Indeed, bacteria that feed on organic matter consume oxygen as part of their metabolism, just as we humans need to both eat and breathe. The product of the decomposition is generally cellular material and carbon dioxide. The more organic matter is present, the more bacteria feed on it, and the greater the oxygen depletion. For this reason, the amount of organic matter is directly related to oxygen depletion, and it is useful to measure the quantity of organic matter not in terms of its own mass but in terms of the mass of oxygen it will have removed by the time it is completely decomposed by bacteria. This quantity is called the *Biochemical Oxygen Demand* and noted BOD. Like disolved oxygen DO, it is expressed in mg/L. BOD values can be extremely large in comparison to levels of dissolved oxygen. For example, BOD of untreated domestic sewage generally exceeds of 200 mg/L and drops to 20–30 mg/L after treatment in a conventional wastewater treatment facility. Still, a value of 20 mg/L is high in comparison to the maximum, saturated value of dissolved oxygen (no more than 8 to 12 mg/L). This implies that even treated sewage must be diluted, lest it completely depletes the receiving stream from its oxygen.

Should the BOD of a waste be excessive and the DO value reach zero, the absence of oxygen causes an anaerobic condition, in which the oxygen-demanding bacteria die off and are replaced by an entirely different set of non-oxygen-demanding bacteria, called anaerobic bacteria. The by-product of their metabolism is methane (CH_4) and hydrogen sulfide (H_2S), both of which are gases that escape to the atmosphere and of which the latter is malodorous. Needless to say, such condition is to be avoided at all cost!

Under normal, aerobic conditions, organic matter decays at a rate proportional to its amount, that is, the decay rate of BOD is proportional to the BOD value. Thus, we write:

$$\frac{d\ \mathrm{BOD}}{dt} = -K_d\ \mathrm{BOD}, \tag{15.71}$$

where K_d is the decay constant of the organic matter. Since by definition, BOD is the amount of oxygen that is potentially depleted, every milligram of BOD that is decayed entrains a loss of one milligram of dissolved oxygen. Therefore, the accompanying decay of DO is:

$$\frac{d\ \mathrm{DO}}{dt} = -K_d\ \mathrm{BOD}, \tag{15.72}$$

Like the reaeration coefficient, the decay coefficient depends on temperature. The formula most often used is

Table 15.5: Typical values of the decay coefficient for various types of wastes. [From Davis and Cornwell, 1991]

Waste type	K_d at 20°C (in 1/day)
Raw domestic sewage	0.35–0.70
Treated domestic sewage	0.12–0.23
Pollutted river water	0.12–0.23

$$K_d(\text{at } T) \;=\; K_d(\text{at } 20°\text{C})\ 1.047^{T-20}\ , \tag{15.73}$$

where T is here the temperature in degrees Celsius. The value at the reference temperature of 20°C depends on the nature of the waste. Table 15.5 lists a few common values.

Oxygen sag curve

Let us now consider a river in which a BOD-laden discharge is introduced. Downstream of that point, the decay of BOD is accompanied by a consumption of DO, which in turn creates an increasing deficit of dissolved oxygen. But, as the oxygen deficit grows, so does the reaeration rate, according to (15.67). At some point downstream, reaeration is capable of overcoming the loss due to BOD decomposition, which gradually slows down as there is increasingly less BOD remaining. The net result is a variation of dissolved oxygen downstream of the discharge that first decays and then recovers, with a minimum somewhere along the way. Plotting the DO value as a function of the downstream distance yields a so-called oxygen-sag curve.

Because the worst water condition occurs where the dissolved oxygen is at its lowest, it is important to determine the location of the minimum, if any, and its value. For this purpose, let us model the river as a one-dimensional system, with uniform volumetric flowrate Q along the downstream direction x measured from the point of discharge ($x = 0$). The 1D assumption presupposes relatively rapid vertical and transverse mixing of the discharge. Let us further assume that the situation is in steady state (constant discharge and stream properties unchanging over time), and that the flow is sufficiently swift to create a highly advective situation, so that we may neglect diffusion in the downstream direction.

We establish the BOD and DO budgets for a slice dx of the river, as depicted in Figure 15.18. The volume of this slice is $V = Adx$ and the surface exposed to the air is $A_s = Wdx$, where A is the river's cross-sectional area and W its width.

In steady state, there is no accumulation or depletion, and the BOD budget demands that the downstream export be the upstream import minus the local decay, namely:

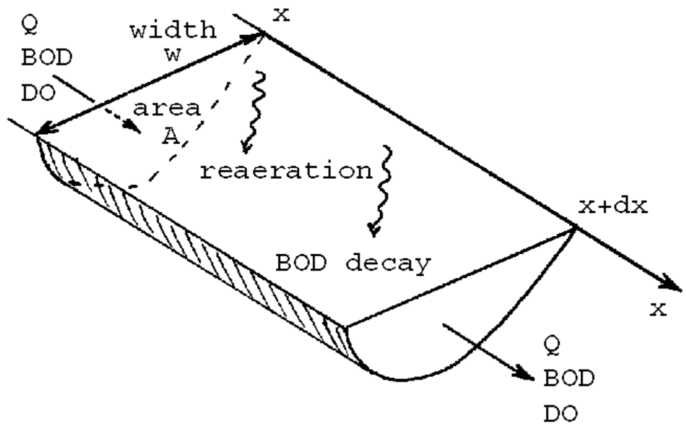

Figure 15.18: Dissolved oxygen and BOD budgets in a stretch of a transversely well mixed river.

$$Q\ \mathrm{BOD}(x+dx) \ = \ Q\ \mathrm{BOD}(x) \ - \ K_d\ V\ \mathrm{BOD}.$$

Using $V = Adx$ and re-arranging, we write:

$$Q\ \frac{\mathrm{BOD}(x+dx) - \mathrm{BOD}(x)}{dx} \ = \ -\ K_d\ A\ \mathrm{BOD}.$$

In the limit of a short slice, the difference on the left-hand side becomes a derivative in x, and since $Q = Au$, a division by A yields:

$$u\ \frac{d}{dx}\mathrm{BOD} \ = \ -\ K_d\ \mathrm{BOD}. \tag{15.74}$$

The solution is

$$\mathrm{BOD}(x) \ = \ \mathrm{BOD}_o\ \exp\left(-\frac{K_d\ x}{u}\right)\ , \tag{15.75}$$

where BOD_o is the value of the biochemical oxygen demand of the waste discharged at $x = 0$.

Similarly, the budget of dissolved oxygen consists in balancing the downstream export plus the local decay with the upstream import and the local reaeration:

$$Q\ DO(x+dx) \ + \ K_d\ V\ BOD \ = \ Q\ DO(x) \ + \ k_r\ A_s\ (DO_s - DO).$$

Using $V = Adx$ and $A_s = Wdx$ and re-arranging the terms, we obtain

$$Q\ \frac{\mathrm{DO}(x+dx) - \mathrm{DO}(x)}{dx} \ = \ k_r W\ (\mathrm{DO}_s - \mathrm{DO}) \ - \ K_d\ A\ \mathrm{BOD}.$$

In the limit of a short slice, the differential equation is:

$$u \frac{d}{dx}\mathrm{DO} \ = \ \frac{k_r W}{A} \ (\mathrm{DO}_s - \mathrm{DO}) \ - \ K_d \ \mathrm{BOD}.$$

Next, we recall $A/W = H$ (the cross-sectional area of the river divided by its width is the average depth) and $k_r/H = K_r$, and we also substitute for BOD the solution given by (15.75):

$$u \frac{d}{dx}\mathrm{DO} \ = \ K_r \ (\mathrm{DO}_s - \mathrm{DO}) \ - \ K_d \ \mathrm{BOD}_o \ \exp\left(-\frac{K_d \ x}{u}\right) \ . \tag{15.76}$$

The solution is

$$\begin{aligned} \mathrm{DO}(x) \ &= \ \frac{K_d \ \mathrm{BOD}_o}{K_d \ - \ K_r} \left[\exp\left(-\frac{K_d \ x}{u}\right) \ - \ \exp\left(-\frac{K_r \ x}{u}\right)\right] \\ &\quad - \ (\mathrm{DO}_s - \mathrm{DO}_o) \ \exp\left(-\frac{K_r \ x}{u}\right) \ + \ \mathrm{DO}_s, \end{aligned} \tag{15.77}$$

where DO_o is the level of dissolved oxygen at the discharge point, which may or may not be equal to the saturated value DO_s. The first term represents the effect of the BOD consumption, while the second represents the recovery toward saturation from a possible prior deficit.

As anticipated earlier, the function $\mathrm{DO}(x)$ may reach a minimum (Figure 15.19). Setting the derivative of DO with respect to x equal to zero and solving for the critical value x_c, we obtain:

$$x_c \ = \ \frac{u}{K_r - K_d} \ \ln\left\{\frac{K_r}{K_d}\left[1 \ - \ \frac{(K_r - K_d)(\mathrm{DO}_s - \mathrm{DO}_o)}{K_d \ \mathrm{BOD}_o}\right]\right\}. \tag{15.78}$$

This is the distance downstream from the discharge to the location where the lowest dissolved oxygen occurs. At that location, the BOD decay rate exactly balances the reaeration rate, so that there is no local change in the amount of dissolved oxygen. Note that an x_c value may not exist if the expression inside the logarithm is negative. This occurs when the upstream oxygen deficit $\mathrm{DO}_s - \mathrm{DO}_o$ is relatively large compared to the BOD_o loading, in which case the dissolved oxygen simply recovers from its initial deficit without passing through a minimum anywhere downstream.

There is a useful simplification in the case when the stream has no prior oxygen deficit ($\mathrm{DO}_s - \mathrm{DO}_o = 0$). The expression for the critical distance reduces to:

$$x_c \ = \ \frac{u}{K_r - K_d} \ \ln\left(\frac{K_r}{K_d}\right), \tag{15.79}$$

which, we note, is independent of the loading BOD_o and always exists. The ratio K_r/K_d has been called the *self-purification* ratio.

Once the critical distance x_c is determined, the minimum value $\mathrm{DO}_{\mathrm{min}}$ of the dissolved oxygen is found by substitution of (15.78) or (15.79), whichever applies, into (15.77). No mathematical expression is written down here because it is extremely cumbersome. In practice, numerical values are used before the substitution.

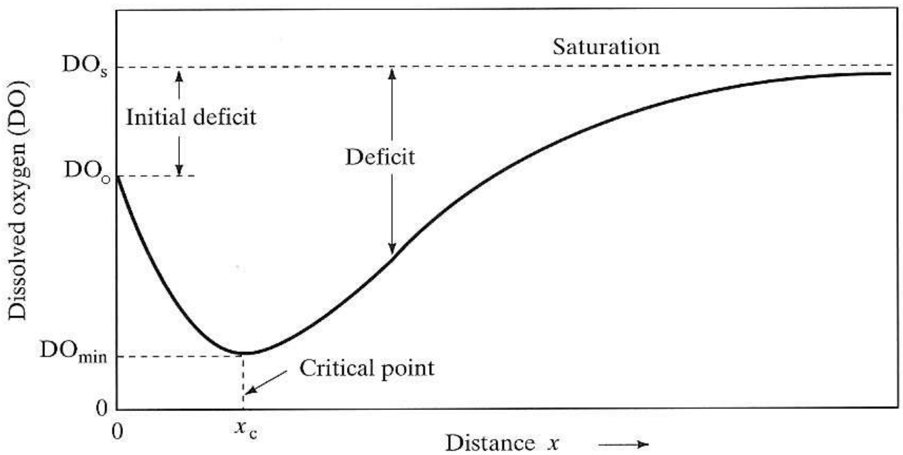

Figure 15.19: The oxygen-sag curve showing the initial decay of dissolved oxygen under pollutant loading and subsequent recovery by reaeration. (Figure adapted from Masters, 1997)

Mathematically, it may happen that DO_{min} falls below zero, which is physically impossible. Should this be the case, the dissolved oxygen reaches zero before a minimum is reached [at an x location found by setting expression (15.77) to zero], and DO = 0 exists further downstream. Over this stretch of the stream, the BOD no longer decays according to (15.74) because there is not enough oxygen, and the preceding formalism no longer holds. Instead, anaerobic degradation must be considered.

The model tacitly also assumes that the only oxygen demands on the river are the BOD of the discharge and any prior oxygen deficit. In actual rivers, sediments may cause a significant additional oxygen demand, because many forms of river pollution contain suspended solids (SS) that gradually settle along the river bed, spreading over a long distance, and subsequently decay. In heavily polluted rivers, this sediment oxygen demand (SOD) can be in the range 5–10 mg/(m^2.day) along the surface of the channel bed. In budget (15.76), the sediment oxygen demand appears as a sink term on the left-hand side equal to $-SOD/h$, and solution (15.77) needs to be amended, but this is beyond our scope.

15.10 Sedimentation and Erosion

Rivers and stream carry material in the form of solid particles that may alternatively be deposited on the river bed (sedimentation) and entrained into the moving water (erosion). Such material may be contaminated, and therefore one pollution transport mechanism in a river is by successive erosion and sedimentation.

Table 15.6: Typical diameters of sediment particles

Type of particle	d_s (mm)
Fine clay	smaller than 0.001
Medium clay	between 0.001 and 0.002
Coarse clay	between 0.002 and 0.004
Fine silt	between 0.004 and 0.016
Medium silt	between 0.016 and 0.031
Coarse silt	between 0.031 and 0.062
Fine sand	between 0.062 and 0.25
Medium sand	between 0.25 and 0.50
Coarse sand	between 0.50 and 2.0
Fine gravel	between 2.0 and 8.0
Medium gravel	between 8 and 16
Coarse gravel	between 16 and 64
Small cobble	between 64 and 128
Large cobble	between 128 and 256
Small boulder	between 256 and 512
Medium boulder	between 512 and 1024
Large boulder	larger than 1024

Studies have shown that the entrainment of a solid particle lying on the bed into the flow depends primarily on the size of the particle and the stress of the moving water onto the bed. Physically, the bottom stress exerts on a particle lying on top of the packed bed a force that is a combination of drag and lift and, depending on the particle's weight, this force may or may not be sufficient to entrain the particle. A first quantity to consider, therefore, is the apparent weight of the particle in the water (actual weight corrected by the buoyancy force):

$$\begin{aligned}\text{Apparent weight of particle} &= \text{Actual weight} - \text{Weight of displaced water} \\ &= \rho_s \left(\frac{\pi d_s^3}{6}\right) g - \rho \left(\frac{\pi d_s^3}{6}\right) g \\ &= \frac{\pi}{6} (\rho_s - \rho)\, g d_s^3, \end{aligned} \tag{15.80}$$

where d_s is the particle diameter, ρ_s its density[7], and ρ the density of water. The particle is assumed to be spherical. Typically, sediment particles consist of quartz and clay minerals with a density of

$$\rho_s = 2650 \text{ kg/m}^3. \tag{15.81}$$

A particle is entrained into the flow when the bottom stress τ_b exceeds a critical value. The greater the weight of the particle, the stronger must be the stress. Ac-

[7] The subscript s stands for *solid*.

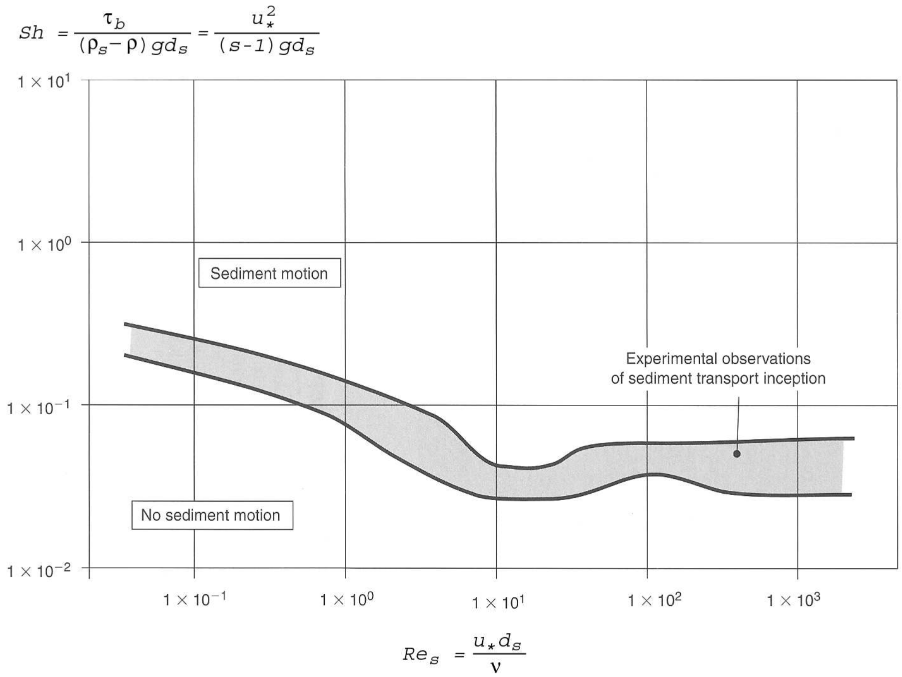

Figure 15.20: Relation between the particle size of the bed material and the Shields parameter, which compares the bottom stress to the resistance to particle entrainment. Note the logarithmic scales and that the horizontal axis is the Reynolds number at the size of the particle, with $u_* = \sqrt{\tau_b/\rho}$ being the turbulent velocity and ν the kinematic viscosity (1.01×10^{-6} m^2/s for water at ambient temperatures). [Adapted from Chanson, 2004]

cording to Equations (15.8) and (15.15), the bottom stress is related to the bed slope S by

$$\tau_b = \rho g R_h S, \tag{15.82}$$

where R_h is the hydraulic radius. For a wide and shallow river, R_h is nearly equal to the water depth h, and $\tau_b \simeq \rho g h S$, which is a more practical quantity because depth is much easier to determine than the hydraulic radius.

Over the cross-sectional area $A_s = \pi d_s^2/4$ of a particle of diameter d_s, this force is on the order of $\tau_b A_s \sim \tau_b d_s^2$, and it is to be compared to the apparent weight of the particle (proportional to d_s^3), given by (15.80). The ratio defines a dimensionless number, called the *Shields parameter*

$$Sh = \frac{\tau_b}{(\rho_s - \rho) g d_s} = \frac{u_*^2}{(s-1) g d_s}, \tag{15.83}$$

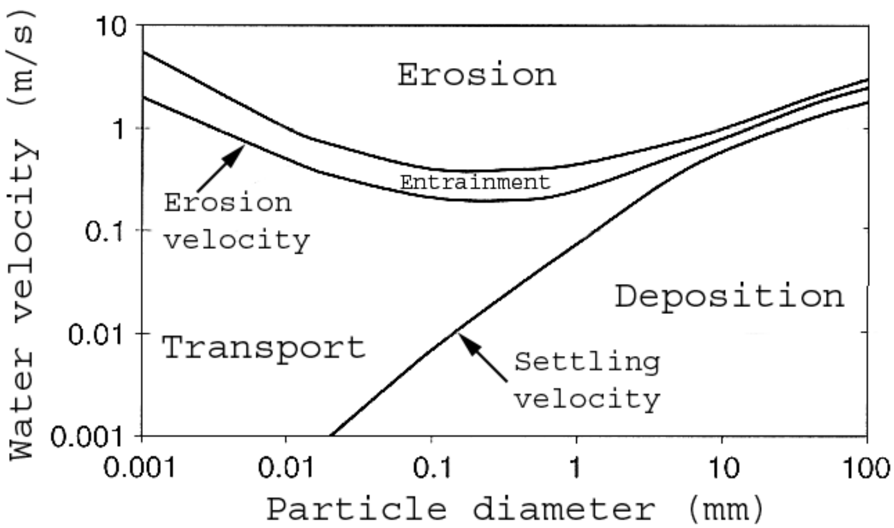

Figure 15.21: Hjulstrom diagram relating flow velocity and bed material size to erosion, entrainment, transport and deposition. (From Ward and Trimble, 2004)

in the definition of which numerical constants have been eliminated for simplification, u_* is the turbulent velocity defined from the bottom stress by (See Section 8.2)

$$u_* = \sqrt{\frac{\tau_b}{\rho}}, \tag{15.84}$$

and $s = \rho_s/\rho = 2.65$ is the particle specific gravity. Figure 15.20 shows how the value of the Shields parameter required to lift a particle into the flow depends on the particle diameter d_s. The relationship is not unique and there is some scatter, because underlying factors are present, such as the particle shape and possible cohesion forces among particles. Most particles fall in the asymptotic regime (right side of graph) for which the middle value is $Sh_{crit} = 0.047$. With this critical value and for $s = 2.65$, the entrainment criterion can be stated as:

$$\begin{aligned} &\text{If} \quad u_*^2 < 0.078\, gd_s \quad \longrightarrow \quad \text{no entrainment} \\ &\text{If} \quad u_*^2 > 0.078\, gd_s \quad \longrightarrow \quad \text{erosion.} \end{aligned}$$

Because the mean stream velocity u is intimately related to the bottom stress, via Equation (15.8), the Shields diagram of Figure 15.20 can be recast in terms of particle diameter and stream velocity. The result is the so-called *Hjulstrom diagram* (Figure 15.21), which also shows the settling velocity.

Once particle are entrained into the flow, they have a tendency to settle back to the bottom. For a bottom stress only slightly larger than the critical value, particles take off, make a leap and fall back onto the bottom. This process is called *saltation* (= making jumps), but for larger values of the bottom stress, turbulent motions can overcome the particle settling velocity and keep particles aloft and far away from the bottom. Whether particles saltate along the bottom or are mixed throughout the water column depends on how the turbulent velocity u_* compares to the particle settling velocity w_s.

The particle settling velocity is the downward velocity at which the particle falls when its apparent weight is counteracted by the upward drag force, as for a parachute:

$$\frac{\pi}{6}\,(\rho_s - \rho)\,g d_s^3 \;=\; \frac{1}{2}\,C_{Ds}\,\rho w_s^2 \frac{\pi d_s^2}{4}\;, \tag{15.85}$$

where C_{Ds} is the drag coefficient of the fluid flow around the particle. For typical sediment particles, an experimental value for this drag coefficient spanning a wide range of Reynolds numbers was provided by Cheng (1997):

$$C_{Ds} \;=\; \left[\left(\frac{24}{Re_s}\right)^{2/3} \;+\; 1\right]^{3/2}, \tag{15.86}$$

where the Reynolds number at the particle level is defined as

$$Re_s \;=\; \frac{w_s d_s}{\nu}\;. \tag{15.87}$$

Solving (15.85) for the settling velocity, we obtain:

$$w_s \;=\; \sqrt{\frac{4(s-1)gd_s}{3C_{Ds}}}\;. \tag{15.88}$$

Table 18.2 lists values of the drag coefficient and settling velocity for a variety of particle diameters.

Laboratory experiments indicate that suspension in the water column occurs when

$$u_* \;>\; (0.2 \text{ to } 2)\; w_s. \tag{15.89}$$

A modification of the Shields diagram 15.20 that incorporates the suspension criterion is shown in Figure 15.22.

The amount of sediment transported by the stream, if any, is called the *wash load*, *suspended load* or simply *bed load*. The load is carried downstream by a combination of sliding, rolling and bouncing of the particles along the bottom. Several formulae have been proposed over the years to estimate this particle transport. A simple formula for m_s, the particle mass transport per unit width of stream, is due to Nielsen (1992):

$$m_s \;=\; 1.63(Sh - Sh_{crit})\rho_s d_s v_s, \tag{15.90}$$

in which v_s is the average horizontal speed of the particles, taken equal to $4.8u_*$, Sh is the Shields parameter defined in (15.83) and Sh_{crit} is the critical value obtained from

Table 15.7: Drag coefficients and settling velocities of sediment particles in water, based on a specific gravity $s = 2.65$. [From Chanson, 2004]

d_s (mm)	C_{Ds}	w_s (m/s)
0.1	36.2	0.008
0.2	8.0	0.023
0.5	2.4	0.067
1.0	1.6	0.117
2.0	1.3	0.186
5.0	1.1	0.314
10	1.0	0.454
20	1.0	0.650
50	1.0	1.034
100	1.0	1.466
200	1.0	2.075

Figure 15.20 or Figure 15.22. Another, somewhat more complicated formula for the bed load transport is the so-called Meyer–Peter–Muller equation (Ward and Timble, 2004):

$$\frac{m_s}{\rho_s \sqrt{(s-1)gd_s^3}} = \left[\frac{4u_*^2}{(s-1)gd_s} - 0.188 \right]^{3/2}, \tag{15.91}$$

where m_s is expressed in kg/(m·s), d_s is the mean particle size (in m), and $u_* = \sqrt{\tau_b/\rho}$ the turbulent velocity (in m/s). Regardless of the formula being used, the value obtained ought to be considered as very approximate.

Example 16.2

An application here

Problems

16-1. The White River in Vermont (USA) has a channel cross-section resembling a parabola with profile $b(y) = a\ y^2$, where y is the cross-channel variable (defined with $y = 0$ at the center of the stream) and $b(y)$ is the bottom elevation measured from zero at the center. The Manning coefficient is $n = 0.040$ and the downstream bed slope is $S = 5.0 \times 10^{-5}$. In the summer, the width of the river and the center water depth are, respectively, 9.8 m and 1.2 m.

(a) In that season, what are the hydraulic radius of the river, its mean velocity and volumetric discharge?

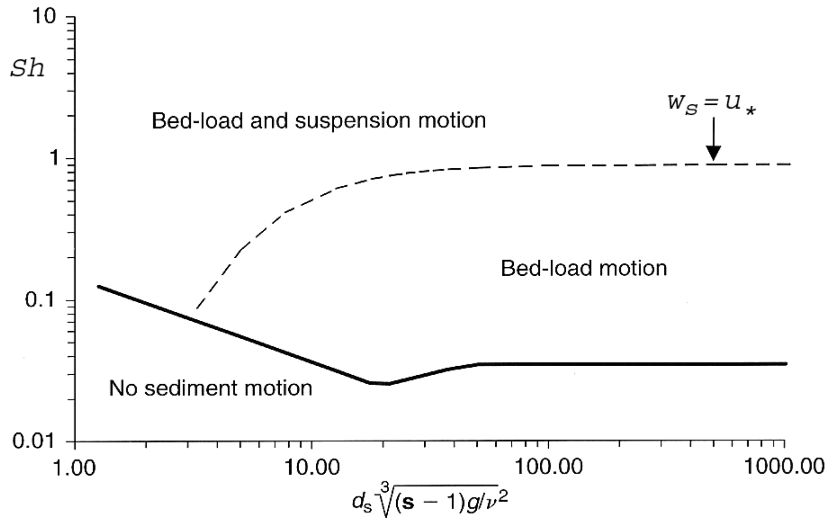

Figure 15.22: Modified Shields diagram showing the critical stress (solid line) and the suspension criterion (dashed line) versus the non-dimensionalized particle diameter. (From Chanson, 2004)

(b) Is the summer flow subcritical or supercritical?

(c) What is the energy dissipation rate per meter of river?

(d) In winter, the discharge is 10 times larger. What are then the center depth and average velocity?

16-2. Determine the water depth h and velocity u in uniform river flow obeying the Manning formula (15.21), each in term of the discharge Q. Assume a wide rectangular channel.

Then apply this to the Rhine River near Karlsruhe in Germany, where the channel width is 171 m, the bed slope 3.13×10^{-4}, and the Manning coefficient $n = 0.022$. If the channel depth is 4.8 m, what maximum discharge can flow through the channel before flooding occurs?

16-3. Somewhere in the Alps, a mountain lake discharges into a 8-m wide stream with boulders across the bottom and gravel along its sides, thus having a Manning coefficient $n = 0.041$. At the starting point on the edge of the lake, the stream bottom lies 0.90 m below the open water level in the lake and, downstream of that point, the bottom slope is uniform at $S = 0.005$.

(a) What is the volumetric discharge of the lake into the stream? And, for this discharge, can the bottom slope be considered mild or steep?

(b) What are the water depth and velocity at the head of the stream? And, what are they far downstream (assuming no change in slope, width or bottom roughness along the way)?

(c) At what value of the lake's open-water level (measured from the stream bottom at its head) would the stream slope switch from mild to steep?

16-4. A 5.2 m wide, flat-bottom stream carries 11.6 m^3/s down a slope that changes quite abruptly from 0.0013 to 0.130. The Manning coefficient n remains the same at 0.035 despite the change in slope.

(a) Assuming that each stretch of slope is fairly long, determine the water depth far upstream, at the knee (point where the slope changes) and far downstream.

(b) At which point is the velocity greatest?

16-5. A 4.2 m wide channel is lined with coarse sand with average particle diameter d_s of 1.35 mm and Manning coefficient n estimated at 0.022. Its slope S is 3.2 $\times$ 10^{-4}. What is the minimum discharge that causes bed erosion?

16-6. The drainage area of Lull's brook at the level of Hartland, Vermont (USA), is 41.9 km^2 and receives an annual precipitation of 1.14 m. Evaporation and seepage through the ground contribute to a water loss of 79%, so that only 21% flows into the stream. The channel width is 1.8 m, bed slope 3.8 $\times$ 10^{-3}, and Manning coefficient 0.04.

(a) What is the average stream discharge? What are the water depth and velocity under the assumption of uniform flow?

(b) Is the slope mild or steep?

(c) What is the bottom stress?

(d) What size particles can this stream keep in suspension? (For this, use the criterion: settling velocity = turbulent velocity.)

16-7. What stream velocity is required to cause a bed load transport of 0.15 kg/(m$cdot$s) when sediment particles have a diameter of 0.06 mm, the bottom roughness (due to ripples) is 0.5 mm and the water depth is 0.85 m?

16-8.

Chapter 16

NEXT CHAPTER

THIS IS TO ENSURE THAT CHAPTER 16 ENDS ON AN EVEN PAGE SO THAT CHAPTER 17 CAN BEGIN ON AN ODD PAGE.